# Analytical verification of scaling laws for the Ising model with external field in fractal lattices

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We use an exact recursion procedure to verify analytically, without any intermediary numerical calculation, the validity of the hyperscaling (Josephson) law extended to fractals, the Rushbrooke and Griffiths scaling laws for the Ising ferromagnet with external magnetic field in the whole family of Migdal-Kadanoff-like hierarchical lattices. [S1063-651X(99)05809-2]

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Many classical spin systems, such as the Ising and Potts models, defined on hierarchical lattices (HLs) constitute a class of exactly solvable models which attracted much attention in the study of critical phenomena. The analytical solutions of spin models on the Bethe lattice, for example, are cited by Baxter [1] as one of the interesting exact solutions for higher-dimensional spin systems. The Bethe lattice can be viewed as a kind of HL [2–5] which is a relevant family of non-Bravais lattices that can be considered, in many situations, as approximated lattices of some Bravais ones (see, e.g., Ref. [6]). Some results are relatively simple to obtain using this kind of fractal lattices, in particular, critical frontiers and correlation length critical exponents. However, the exact calculation of other physical quantities, such as specific heat, magnetization, and susceptibility, as well as their corresponding critical exponents, are much more complicated to obtain within the HL approach and we sometimes find in the literature the use of heuristic recipes to obtain these functions and exponents [7–10].

It is well known that in fractal systems, the critical exponents depend of other geometric parameters, such as connectivity and lacunarity [5], and not only of the fractal dimension itself. Hence, the classification based on universality classes cannot work out on fractal systems in the same way as on translationally invariant lattices. Also the validity or not of the scaling laws between the critical exponents for fractal systems, and the role played by the fractal dimension in these relations, yeld controversial results in the literature. Concerning the hyperscaling law ($d\nu=2-\alpha$), it has been numerically verified in a number of HL systems (with the fractal dimension $d_f$ replacing $d$) [11–13] and has been proved analytically for the three-state antiferromagnetic Potts model on a diamond-type HL family [14] and for the Ising ferromagnet on the whole family of Migdal-Kadanoff-like HLs [15]. Numerical results for the Ising model on Sierpinski carpets [16] verify the hyperscaling only through an effective dimension which is slightly different from the lattice fractal dimension. Concerning the Rushbrooke scaling law ($\alpha+2\beta+\gamma=2$), there is much less evidence in favor of its validity on fractal systems. It has been verified [17] for the Potts ferromagnet on the Wheatstone-bridge HL using approximate methods in the derivation of $\beta$ and $\gamma$. It has also been verified for the Ising ferromagnet in an *m*-sheet Sierpinski gasket family using numerical values for the exponents derived from exact expressions of the thermal quantities [12].

We consider the Ising ferromagnet with external magnetic field on the family of Migdal-Kadanoff-like hierarchical lattices. These lattices are generated in an iterative manner, starting from a two-point lattice joined by a single bond (level $n=0$) which is replaced by a basic cell consisting of $P$ branches in parallel, each of them comprising $b$ bonds in series. This recursive procedure is illustrated in Fig. 1 for the cases ($P=2,b=2$) (the diamond HL) and ($P=2,b=3$). In the $n\rightarrow\infty$ limit one obtains a lattice, which we denote as $\mathrm{HL}_{(P,b)}$, with fractal dimension

$$d_f^{(P,b)}=\frac{\ln Pb}{\ln b}. \tag{1}$$

In the following calculations the paramaters $P\geqslant 2$ and $b\geqslant 2$ are fixed.

The model is described by the dimensionless Hamiltonian

$$-\beta\mathcal{H}_n=K_n\sum_{\langle i,j\rangle}\sigma_i\sigma_j+H_n\sum_{\langle i,j\rangle}(\sigma_i+\sigma_j), \tag{2}$$

where $\beta=1/k_BT$, T being the temperature, $K_n=\beta J_n$, $J_n>0$ is the coupling constant between nearest-neighbor pairs at the $n$ level, $H_n=\beta B_n$, and $B_n$ is the external magnetic

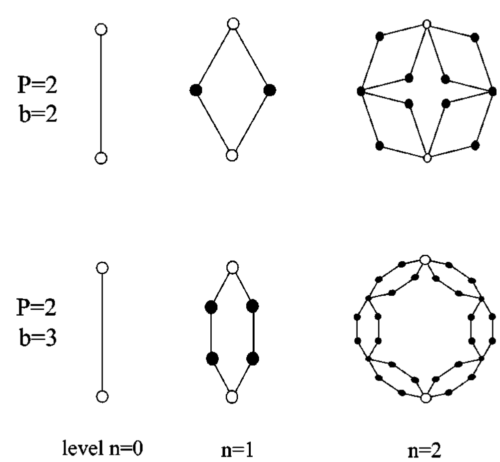


FIG. 1. First three steps of construction of the $\mathrm{HL}_{(P=2,b=2)}$ and $\mathrm{HL}_{(P=2,b=3)}$. The open circles are the root sites of the hierarchical lattices.

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field at level *n*. The sum is over all the first neighbors $\langle i,j\rangle$ of the lattice. Note that this Hamiltonian provides a magnetic field of magnitude $z_i^{(n)}H_n$ at the spin $\sigma_i$, where $z_i^{(n)}$ is the coordination number of site *i* at the *n*-level HL. The presence of this coordination number in the field term asserts that the Hamiltonian (2) is closed (no more couplings among the spins are generated) and form-invariant under our RG transformation [18,5].

In Ref. [15] we considered the zero field model and we shown that the dimensionless internal energy per bond for the *n* level system follows the exact recursion relation

$$E_n^{(b)}=A^{(b)}(x_n)E_{n-1}^{(b)}+C^{(b)}(x_n). \tag{3}$$

with the functions $A^{(b)}$ and $C^{(b)}$ given by (we defined $x \equiv e^K$)

$$A^{(b)}(x)=\frac{4x^2(x^4-1)^{b-1}}{(x^2+1)^{2b}-(x^2-1)^{2b}}$$

and

$$C^{(b)}(x)=\frac{4x^2(x^4-1)}{(x^2+1)^{2b-1}-(x^2-1)^{2b-1}}. \tag{4}$$

Following along this same procedure (described in details in Refs. [19,15]), we obtain here a recurrence for the local magnetizations at different levels given by

$$\sum_{i=1}^{b-1}\langle\sigma_i\rangle^{(b)}=B^{(b)}(x_n,h_n)\{\langle\mu_1\rangle^{(b)}+\langle\mu_2\rangle^{(b)}\}, \tag{5}$$

with (in the particular case $H=0$)

$$B^{(b)}(x,h=1)=\frac{x^4-1}{2}\left(\frac{(x^2+1)^{b-1}-(x^2-1)^{b-1}}{(x^2+1)^b+(x^2-1)^b}\right), \tag{6}$$

where we defined the variable $h\equiv e^H$. We shall omit here the expression of $B^{(b)}(x,h)$ for the general case $H\neq 0$ since it is quite long. In Eq. (5), the spins $\mu_1$ and $\mu_2$ were joined by one bond at the $(n-1)$ level HL, which, at level *n*, was replaced by the basic cell which has in one of its *P* bonds (note that these *P* bonds are equivalent) the $(b-1)$ spins $\sigma$.

From Eq. (5) we can show that the magnetization (per site) of the entire lattice, defined by

$$m_n(x,h)=\frac{1}{N_{cn}^{(P,b)}}\sum_i z_i^{(n)}\sigma_i, \tag{7}$$

where $N_{cn}^{(P,b)}=\Sigma_i z_i^{(n)}=2(bP)^n$ was introduced in order to normalize the magnetization at $T=0$, obeys the relation

$$m_n(x_n,h_n)=\left(\frac{1+2B^{(b)}(x_n,h_n)}{b}\right)m_{n-1}(x_{n-1},h_{n-1}). \tag{8}$$

From Eq. (8) we can show that the magnetization of the system in the limit $n\to\infty$ is given by

$$m(x,h)=\prod_{i=1}^{\infty}\left(\frac{1+2B^{(b)}(x_i,h_i)}{b}\right), \tag{9}$$

which generalizes a previous result [20] obtained for the zero field and $b=2$ case.

Also from Eq. (8) we can show that the zero-field susceptibility defined by

$$\chi_n(x_n)\equiv\left.\frac{\partial m_n(x_n,h_n)}{\partial h_n}\right|_{h_n=1} \tag{10}$$

obeys the recursive relation

$$\chi_n(x_n)=r_h^{(P,b)}\left(\frac{1+2B^{(b)}(x_n,h_n=1)}{b}\right)\chi_{n-1}(x_{n-1}), \tag{11}$$

where we defined

$$r_h^{(P,b)}\equiv\left.\frac{\partial h'(x,h)}{\partial h}\right|_{h_c}=P[1+2B^{(b)}(x,h_c)], \tag{12}$$

and $h'(x,h)$ is the RG transformation which we define in the following.

In order to complete our recursive equations we need the renormalization of the coupling $K_{n-1}=K'$ (or $x_{n-1}=x'$) and field $H_{n-1}=H'$ (or $h_{n-1}=h'$) in terms of the coupling $K_n=K$ (or $x_n=x$) and field $H_n=H$ (or $h_n=h$). This is established in a standard way by preserving the correlation function between the roots of the HL (see, e.g., Ref. [15]). We shall omit here the expressions of $x'(x,h)$ and $h'(x,h)$ since they are quite long. The RG equations, for all $P\geqslant 2$ and $b\geqslant 2$, admit two trivial stable fixed points (note that for the general case these points are over the axis $h=1$), namely, $x=1$ $(T\to\infty)$ (paramagnetic phase) and $x\to\infty$ $(T=0)$ (ferromagnetic phase), as well as a critical (unstable) fixed point $x^*_{(P,b)}$ $(0<x^*_{(P,b)}<\infty)$. Hereafter we will use the abbreviated notation $(x_c,h_c)$ for the critical point $(x^*_{(P,b)},h=1)$ of the $\mathrm{HL}_{(P,b)}$ system. Linearization of $x'(x)$ in the neighborhood of the critical point $x_c$ leads to the thermal (correlation length) critical exponent $\nu^{(P,b)}$:

$$\nu^{(P,b)}=\frac{\ln b}{\ln r_x^{(P,b)}},\quad\text{where}\quad r_x^{(P,b)}\equiv\left.\frac{dx'(x)}{dx}\right|_{x_c}=PbA^{(b)}(x_c). \tag{13}$$

In Ref. [15] we verified analytically the hyperscaling law extended to fractal systems, namely,

$$d_f^{(P,b)}\nu^{(P,b)}=2-\alpha^{(P,b)}, \tag{14}$$

with

$$\alpha=1+\frac{\ln A^{(b)}(x_c)}{\ln r_x^{(P,b)}}, \tag{15}$$

and using the expressions of Eqs. (1) and (13) also.

Following along the same lines, assuming that close to $x_c$, the magnetization $m_n$ can be written as $m_n=\lambda|\epsilon_n|^{\beta}$, we obtain, from Eq. (8),

$$\beta^{(P,b)} = -\frac{\ln[r_h^{(P,b)}/(Pb)]}{\ln r_x^{(P,b)}}. \tag{16}$$

Analogously, from Eq. (11), we can show that the $\gamma$ exponent of the zero-field susceptibility $[\chi_n = \lambda(\epsilon_n)^{-\gamma}]$ is given by

$$\gamma^{(P,b)} = \frac{\ln[(r_h^{(P,b)})^2/(Pb)]}{\ln r_x^{(P,b)}}, \tag{17}$$

which, taking into account Eqs. (15) and (16) verifies analytically the Rushbrooke scaling law

$$\alpha^{(P,b)} + 2\beta^{(P,b)} + \gamma^{(P,b)} = 2. \tag{18}$$

Assuming that at $x_c$ the system magnetization behaves as $m_n = \lambda(h_n - 1)^{1/\delta}$, we obtain, from Eq. (8),

$$\delta^{(P,b)} = -\frac{\ln r_h^{(P,b)}}{\ln[r_h^{(P,b)}/(bP)]}, \tag{19}$$

from which we can verify [using Eqs. (15) and (16) also] the Griffiths scaling law

$$\alpha^{(P,b)} + \beta^{(P,b)}(\delta^{(P,b)} + 1) = 2. \tag{20}$$

If we also assume the validity of the Fisher scaling law

$$(2 - \eta^{(P,b)})\nu^{(P,b)} = \gamma^{(P,b)}, \tag{21}$$

which relates the $\eta$ exponent of the two-point correlation function $[G(x_c, h_c) \sim r^{-(d-2+\eta)}]$ with the exponents $\nu$ and $\gamma$, we obtain, using Eqs. (13) and (17),

$$\eta^{(P,b)} = \frac{1}{2}\frac{\ln Pb^3}{\ln r_h^{(P,b)}}. \tag{22}$$

In summary, we used a method that allowed us to calculate exact recurrence relations for several thermal quantities of the Ising ferromagnet with external field in the whole family of Migdal-Kadanoff-like hierarchical lattices and to obtain the critical exponents $\nu$, $\alpha$, $\beta$, $\gamma$, and $\delta$. With the exact expressions of these exponents we proved analytically, without any intermediary numerical calculation, the validity of the hyperscaling, Rushbrooke and Griffiths scaling laws for this large class of fractal systems. Assuming also the validity of the Fisher scaling law $[(2-\eta)\nu = \gamma]$ we obtained the exponent $\eta$ of the correlation function for these systems.

As it is well known, the scaling laws are an immediate outcome of the renormalization group in the absence of dangerous irrelevant variables. However, the point of this paper was to provide an explicit, analytical verification of these laws, something that has not been done explicitly earlier for this large class of systems and, mainly, for this complete set of scaling laws.


[1] R. J. Baxter, in *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[2] Y. Gefen, A. Aharony, and B. B. Mandelbrot, Phys. Rev. Lett. **45**, 855 (1980).

[3] Y. Gefen, A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, Phys. Rev. Lett. **47**, 1771 (1981).

[4] R. B. Griffiths and M. Kaufman, Phys. Rev. B **26**, 5022 (1982).

[5] J. R. Melrose, J. Phys. A **16**, 3077 (1983).

[6] A. N. Berker and S. Ostlund, J. Phys. C **12**, 4961 (1979).

[7] H. O. Mártin and C. Tsallis, J. Phys. C **14**, 5645 (1981).

[8] E. M. F. Curado, C. Tsallis, S. V. F. Levy, and M. J. Oliveira, Phys. Rev. B **23**, 1419 (1981).

[9] M. Kaufman and R. B. Griffiths, Phys. Rev. B **28**, 3864 (1983).

[10] M. Kaufman and K. K. Mon, Phys. Rev. B **29**, 1451 (1984).

[11] S. Coutinho, O. D. Neto, J. R. L. de Almeida, E. M. F. Curado, and W. A. M. Morgado, Physica A **185**, 271 (1992).

[12] J. A. Redinz and A. C. N. de Magalhães, Phys. Rev. B **51**, 2930 (1995).

[13] L. da Silva, E. M. F. Curado, S. Coutinho, and W. A. M. Morgado, Phys. Rev. B **53**, 6345 (1996).

[14] J. A. Redinz, A. C. N. de Magalhães, and E. M. F. Curado, Phys. Rev. B **49**, 6689 (1994).

[15] J. A. Redinz, J. Phys. A **31**, 6921 (1998).

[16] P. Monceau, M. Perreau, and F. Hébert, Phys. Rev. B **58**, 6386 (1998).

[17] E. P. da Silva and C. Tsallis, Physica A **167**, 347 (1990).

[18] J. M. Yeomans and M. E. Fisher, Phys. Rev. B **24**, 2825 (1981).

[19] W. A. M. Morgado, S. Coutinho, and E. M. F. Curado, J. Stat. Phys. **61**, 913 (1991).

[20] P. M. Bleher and E. Zalys, Commun. Math. Phys. **120**, 409 (1989).